NEW FORMAT

AJ Sadler

Mathematics Specialist

Student Book

Units 3 & 4

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A J Sadler

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PREFACE

This text targets Units Three and Four of the West Australian course *Mathematics Specialist*, a course that is organised into four units altogether, the first two for year eleven and the last two for year twelve.

This West Australian course, *Mathematics Specialist*, is based on the Australian Curriculum Senior Secondary course *Specialist Mathematics*. With only very slight differences between the Unit Three content of these two courses this text is also suitable for anyone following Unit Three of the Australian Curriculum course, *Specialist Mathematics*. For Unit Four, the West Australian course omits some topics that are in the Australian Curriculum, these being the inverse trigonometric functions, integration by parts, exponential random variables, and some applications involving force. Integration by parts is included in this text as an extension activity and the companion volume for *Mathematics Methods* Unit Four includes some questions involving exponential random variables. The inverse trigonometric functions feature in this text in one question of a Miscellaneous Exercise. The applications involving force are not included in this text.

The book contains text, examples and exercises containing many carefully graded questions. A student who studies the appropriate text and relevant examples should make good progress with the exercise that follows.

Each unit commences with a section entitled **Preliminary work**. This section briefly outlines work of particular relevance to the unit and that students should either already have some familiarity with from the mathematics studied in earlier years, or for which the brief outline included in the section may be sufficient to bring the understanding of the concept up to the necessary level.

A **Miscellaneous exercise** features at the end of each chapter and includes questions involving work from preceding chapters, and from the *Preliminary work* section.

A few chapters commence with, or contain, a '**Situation**' or two for students to consider. Answers to these situations are generally not included in the book. Students should be encouraged to discuss their solutions and answers to these situations.

At times in this series of books I have found it appropriate to go a little beyond the confines of the syllabus for the unit involved. In this regard, when considering the absolute value function, and to meet the syllabus requirement to 'use and apply' $|x|$ and the graph of $y = |x|$, I include solving equations and inequalities involving absolute values. With vectors, when considering whether two moving objects meet, or whether their paths cross, I also consider the closest approach situation. In Unit Four, when using a substitution to integrate expressions, I do not limit such considerations to expressions of the form $f(g(x))g'(x)$. For integration by numerical methods I mention the Trapezium rule and Simpson's rule and when considering differential equations I include Euler's method. With sampling I include mention of a significant difference at the 5% level.

I leave it up to the readers, and teachers, to decide whether to cover these aspects.

Alan Sadler

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IMPORTANT NOTE

This series of texts has been written based on my interpretation of the appropriate *Mathematics Specialist* syllabus documents as they stand at the time of writing. It is likely that as time progresses some points of interpretation will become clarified and perhaps even some changes could be made to the original syllabus. I urge teachers of the *Mathematics Specialist* course, and students following the course, to check with the appropriate curriculum authority to make themselves aware of the latest version of the syllabus current at the time they are studying the course.

Acknowledgements

As with all of my previous books I am again indebted to my wife, Rosemary, for her assistance, encouragement and help at every stage.

To my three beautiful daughters, Rosalyn, Jennifer and Donelle, thank you for the continued understanding you show when I am 'still doing sums' and for the love and belief you show.

To the delightfully supportive team at Cengage – I thank you all.

Alan Sadler

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Mathematics Specialist

Unit Three

3

UNIT THREE PRELIMINARY WORK

This book assumes that you are already familiar with a number of mathematical ideas from your mathematical studies in earlier years.

This section outlines the ideas which are of particular relevance to Unit Three of the *Mathematics Specialist* course and for which familiarity will be assumed, or for which the brief explanation given here may be sufficient to bring your understanding of the concept up to the necessary level.

Read this 'Preliminary work' section and if anything is not familiar to you, and you don't understand the brief mention or explanation given here, you may need to do some further reading to bring your understanding of those concepts up to an appropriate level for this unit. (If you do understand the work but feel somewhat 'rusty' with regards to applying the ideas, some of the chapters afford further opportunities for revision, as do some of the questions in the miscellaneous exercises at the end of chapters.)

- Chapters in this book will continue some of the topics from this preliminary work by building on the assumed familiarity with the work.
- The **miscellaneous exercises** that feature at the end of each chapter may include questions requiring an understanding of the topics briefly explained here.
- Much of the content of this preliminary work section is from Units One and Two of the *Mathematics Specialist* course. It is assumed that students embarking on this Unit Three of *Mathematics Specialist* will have successfully completed these two prior units, will also have successfully completed Units One and Two of *Mathematics Methods* and will also be taking Unit Three of *Mathematics Methods*, probably at the same time as studying this unit.

Number

In the real number system, $\mathbb{R}$, it is assumed that you are familiar with, and competent in the use of, positive and negative numbers, recurring decimals, square roots and cube roots and that you are able to choose levels of accuracy to suit contexts and distinguish between exact values, approximations and estimates.

Numbers expressed with positive, negative and fractional powers should also be familiar to you.

An ability to simplify expressions involving square roots is also assumed.

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The absolute value

The absolute value of a number is the distance on the number line that the number is from the origin.

The absolute value of x is written $|x|$ and equals x when x is positive,
and equals $-x$ when x is negative.

Thus $|3| = 3$, $|-3| = 3$, $|4| = 4$, $|-4| = 4$.

Just as $|x|$ is the distance x is from the origin, $|x - a|$ tells us the distance x is from the number a.

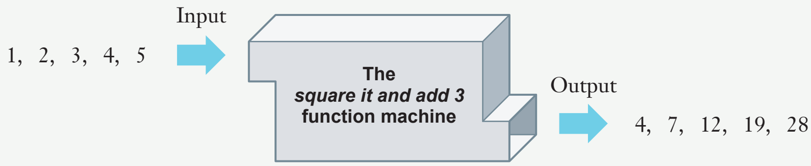
Function

It is assumed that you are familiar with the idea that in mathematics any rule that takes any input value that it can cope with and assigns to it a one, *and only one*, output value is called a **function**.

Familiarity with the function notation $f(x)$ is also assumed.

It can be useful at times to consider a function as a machine. A box of numbers (the **domain**) is fed into the machine, a certain rule is applied to each number, and the resulting output forms a new box of numbers, the **range**.

In this way $f(x) = x^2 + 3$, with domain $\{1, 2, 3, 4, 5\}$, could be 'pictured' as follows:



If we are not given a specific domain, we assume it to be all the real numbers that the function can cope with, sometimes referred to as the **natural domain** of the function.

Thus the function $f(x) = \sqrt{x - 3}$ has a domain of all the real numbers greater than or equal to 3, i.e., $\{x \in \mathbb{R}: x \geq 3\}$.

For this domain the function can put out all the real numbers greater than or equal to zero. Thus the range of the function will be all real numbers greater than or equal to 0, i.e., $\{y \in \mathbb{R}: y \geq 0\}$.

It is assumed you are particularly familiar with linear and quadratic functions, their characteristic equations and their graphs, and have some familiarity with the graphs of $y = x^3$, $y = \sqrt{x}$ and $y = \frac{1}{x}$.

It is further assumed that how the graph of $y = af[b(x - c)] + d$ changes as the values of a, b, c and d are altered is something you have previously considered for various functions.

Remember that linear and quadratic functions are members of the larger family of functions called **polynomial functions**. These are functions of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer and $a_n, a_{n-1}, a_{n-2}, \ldots$ are all numbers, called the **coefficients** of x^n, x^{n-1}, x^{n-2}, etc.

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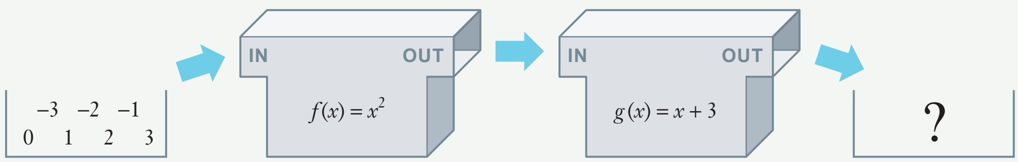
The highest power of x is the **order** of the polynomial.

Thus linear functions, $y = mx + c$, are polynomials of order 1,
quadratic functions, $y = ax^2 + bx + c$, are polynomials of order 2,
cubic functions, $y = ax^3 + bx^2 + cx + d$, are polynomials of order 3, etc.

Though not an idea you would necessarily be familiar with, but one that should seem reasonable, is that of using the output from one function as the input of a second function. In this way we form a **composite function**, also referred to as a **function of a function**.

Suppose that $f(x) = x^2$ and $g(x) = x + 3$.

If we feed the set of numbers $\{-3, -2, -1, 0, 1, 2, 3\}$ into f and then feed the output into g what numbers will g output?



With the domain stated, combining the functions f and g in this way will give a final output of $\{3, 4, 7, 12\}$:

$$\{-3, -2, -1, 0, 1, 2, 3\} \xrightarrow{f(x)} \{0, 1, 4, 9\} \xrightarrow{g(x)} \{3, 4, 7, 12\}$$

We write this combined function as	$g[f(x)]$			
or as	$g \text{ o } f(x)$	or	$g \circ f(x)$	for 'g of f of x'
or as	$gf(x)$.			

Note that though our 'machine diagram' above shows the 'f function' first we write the combined function as $gf(x)$. This is to show that the 'f function', being closest to the '(x)', operates on the x values first.

Algebra

It is assumed that you are already familiar with manipulating algebraic expressions, in particular:

- Expanding and simplifying:

For example	$4(x + 3) - 3(x + 2)$	expands to	$4x + 12 - 3x - 6$
		which simplifies to	$x + 6$
	$(x - 7)(x + 1)$	expands to	$x^2 + 1x - 7x - 7$
		which simplifies to	$x^2 - 6x - 7$
	$(2x - 7)^2$, i.e. $(2x - 7)(2x - 7)$	expands to	$4x^2 - 28x + 49$

- Factorising:

For example	$21x + 7$	factorises to	$7(3x + 1)$
	$15apy + 12pyz - 6apq$	factorises to	$3p(5ay + 4yz - 2aq)$
	$x^2 - 6x - 7$	factorises to	$(x - 7)(x + 1)$
	$x^2 - 9$	factorises to	$(x - 3)(x + 3)$

the last one being an example of the *difference of two squares* result:

$x^2 - y^2$	factorises to	$(x - y)(x + y)$

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- Solving equations

  In particular: linear equations, simultaneous equations,
  quadratic equations, including use of the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,
  exponential equations (e.g. $2^x + 3 = 35$),
  trigonometrical equations (e.g. $\sin x = 0.5$ for $0 \le x \le 360°$),

  and in the use of your calculator to solve equations.

- Completing the square

  To express $x^2 - 6x + 10$, for example, in 'completed square form':

  create a gap to allow the square of half the coefficient of x to be inserted,

	$x^2 - 6x + 10$	$=$	$x^2 - 6x \quad + 10$
insert (and subtract)		$=$	$x^2 - 6x + 9 + 10 - 9$
factorise		$=$	$(x-3)^2 + 1$

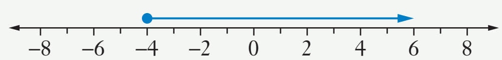
- Inequalities

  It is assumed that you are familiar with the meaning and use of the symbols <, ≤, > and ≥, can solve simple linear inequalities and display the solutions as points on a number line. For example:

  Given
  $$\begin{aligned} 5x - 3 &< 7 \\ 5x &< 10 \\ x &< 2 \end{aligned}$$

  Given
  $$\begin{aligned} 1 - 3x &\le 13 \\ -3x &\le 12 \\ 3x &\ge -12 \\ x &\ge -4 \end{aligned}$$

  Note especially
  - in the example above right, how multiplying by –1 'reverses' the inequality
  - the significance of the open and filled circles.

Vectors

Quantities which have magnitude and direction are called **vectors**.

Common examples of vector quantities are:

Displacement, e.g. 5 km south.
Force, e.g. 6 Newtons upwards.
Velocity, e.g. 5 m/s north.
Acceleration, e.g. 5 m/s^2 east.

Quantities which have magnitude only are not vectors. Such quantities are called **scalars**. Common examples of scalar quantities are:

Distance, e.g. 5 km.
The magnitude of a force, e.g. 6 Newtons.
The magnitude of the acceleration, e.g. 5 m/s^2.
Speed, e.g. 5 m/s.
Energy, e.g. 50 joules.

We represent vector quantities diagrammatically by a line segment in the given direction and whose length represents the magnitude of the vector.

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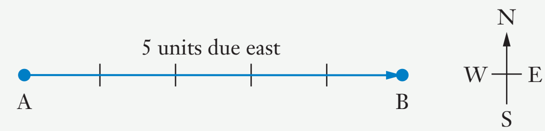
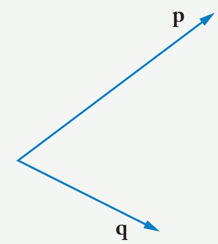
For example, the diagram on the right shows a vector of magnitude 5 units and in direction due east.

We write this vector as $\underline{AB}$ or $\overrightarrow{AB}$.



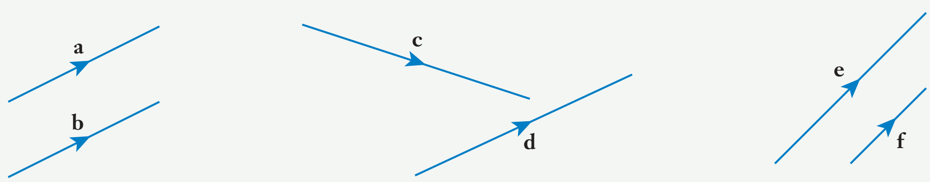
The order of the letters indicates the direction, i.e. **from** A **to** B.

Vectors are also written using lowercase letters and in such cases bold type or underlining is used.



For the magnitude of vector **a**, we write $|\mathbf{a}|$ or $|\underline{a}|$ or simply a.

Vectors are equal if they have the same magnitude **and** the same direction.



a = b
Same magnitude,
same direction.

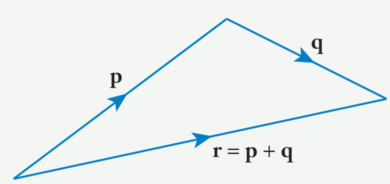
c ≠ d
Same magnitude,
different directions.

e ≠ f
Same direction,
different magnitudes.

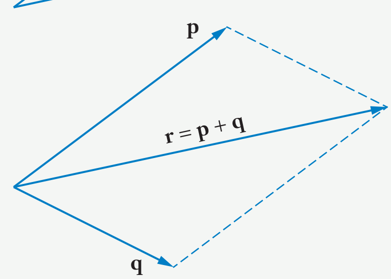
If for some positive scalar k, $\mathbf{b} = k\mathbf{a}$, then **b** is in the same direction as **a** and k times the magnitude.
If k is negative then **b** will be in the opposite direction to **a** and $|k|$ times the magnitude.

Two vectors are parallel if one is a scalar multiple of the other.
If the scalar multiple is positive the vectors are said to be *like* parallel vectors, i.e. in the *same* direction.
If the scalar multiple is negative the vectors are said to be *unlike* parallel vectors, i.e. in *opposite* directions.

To add two vectors means to find the single, or **resultant**, vector that could replace the two. We add the vectors using a **vector triangle** in which the two vectors to be added follow 'nose to tail' and form two of the sides of the triangle. The resultant is then the third side of the triangle with its direction 'around the triangle' being in the opposite sense to the other two.



This vector addition is sometimes referred to as the **parallelogram law**.



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To give meaning to vector subtraction we consider $\mathbf{a} - \mathbf{b}$ as $\mathbf{a} + (-\mathbf{b})$ and then use our technique for adding vectors.

If we add a vector to the negative of itself we obtain the **zero vector**.
The zero vector has zero magnitude and an undefined direction.

Given the vector statement $h\mathbf{a} = k\mathbf{b}$ we can conclude that some scalar multiple of **a** has the same magnitude and direction as some scalar multiple of **b**. If this is the case then either

a and **b** are parallel (because one is a scalar multiple of the other),

or $h = k = 0$.

Thus if **a** and **b** are not parallel and we have a vector expression of the form

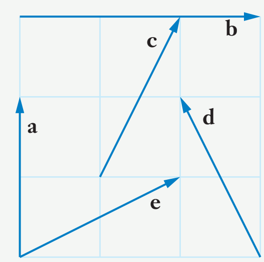
$$p\mathbf{a} + q\mathbf{b} = r\mathbf{a} + s\mathbf{b} \quad \text{(for scalar } p, q, r \text{ and } s) \qquad \leftarrow \text{equation [1]}$$

i.e. $$(p - r)\mathbf{a} = (s - q)\mathbf{b},$$

then $$p = r \quad \text{and} \quad s = q.$$

Thus in equation [1], with **a** and **b** not parallel, we can equate the coefficients of **a**, and we can equate the coefficients of **b**.

Vectors can be expressed in terms of their **horizontal and vertical components**. With **i** and **j** representing unit vectors horizontally and vertically respectively, the vectors **a**, **b**, **c**, **d** and **e** shown in the diagram can be written:

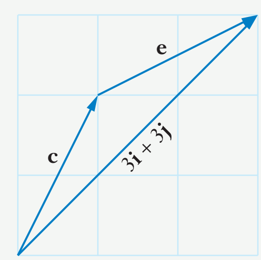


$\mathbf{a} = 2\mathbf{j}$ $\qquad$ $\mathbf{b} = 3\mathbf{i}$

$\mathbf{c} = \mathbf{i} + 2\mathbf{j}$ $\qquad$ $\mathbf{d} = -\mathbf{i} + 2\mathbf{j}$

$\mathbf{e} = 2\mathbf{i} + \mathbf{j}$

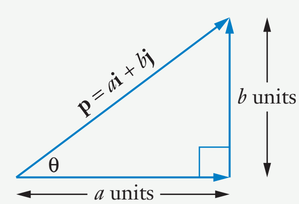
Expressed in this component form the vector arithmetic is straightforward:



$$\begin{aligned} \mathbf{c} + \mathbf{e} &= (\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} + \mathbf{j}) \\ &= 3\mathbf{i} + 3\mathbf{j} \end{aligned}$$

With $$\mathbf{p} = a\mathbf{i} + b\mathbf{j}$$

the **magnitude**, or **modulus**, of **p** is given by



$$|\mathbf{p}| = \sqrt{a^2 + b^2}$$

and θ, see diagram, is found using $\tan\theta = \dfrac{b}{a}$.

The vector $a\mathbf{i} + b\mathbf{j}$ is sometimes written as an **ordered pair** (a, b),
or perhaps $\langle a, b \rangle$,

and sometimes as a **column matrix** $\begin{pmatrix} a \\ b \end{pmatrix}$.

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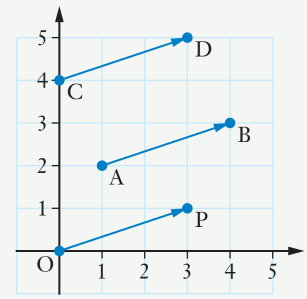
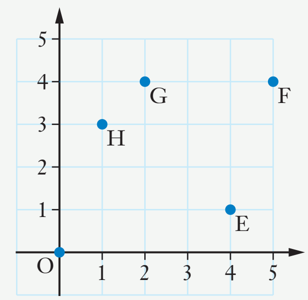
In the diagram on the right $\overrightarrow{OP}$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are each $3\mathbf{i} + \mathbf{j}$. Such vectors are sometimes referred to as **displacement vectors** as they give the *displacement* of P from O, B from A and D from C respectively.



However with O as the origin, only point P has the **position vector** $3\mathbf{i} + \mathbf{j}$.

Point A has position vector $\mathbf{i} + 2\mathbf{j}$,
point B has position vector $4\mathbf{i} + 3\mathbf{j}$,
point C has position vector $4\mathbf{j}$,
point D has position vector $3\mathbf{i} + 5\mathbf{j}$.

Position vectors give the position of a point relative to the origin. Hence the position vector of point H on the right is $\mathbf{i} + 3\mathbf{j}$.



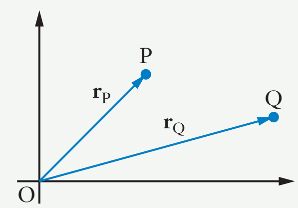
If instead we want to give the position of H **relative to** point E we write:

$$ {}_H\mathbf{r}_E = \overrightarrow{EH} = -3\mathbf{i} + 2\mathbf{j}. $$

Similarly $\quad {}_G\mathbf{r}_E = -2\mathbf{i} + 3\mathbf{j}, \quad {}_F\mathbf{r}_H = 4\mathbf{i} + \mathbf{j}, \quad {}_E\mathbf{r}_G = 2\mathbf{i} - 3\mathbf{j}.$

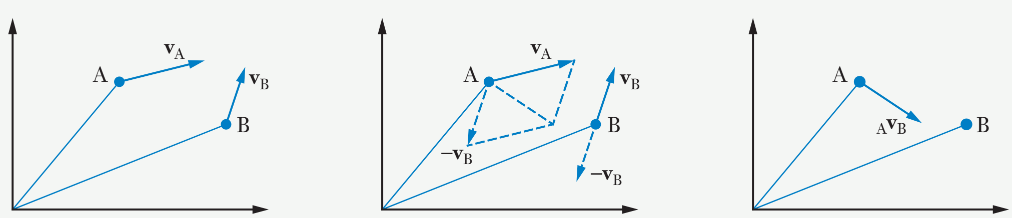
It follows that $\quad {}_P\mathbf{r}_Q = \overrightarrow{QP} = -\mathbf{r}_Q + \mathbf{r}_P$

i.e $\quad {}_P\mathbf{r}_Q = \mathbf{r}_P - \mathbf{r}_Q$



Similarly we can talk of ${}_A\mathbf{v}_B$, the **velocity of A relative to B**. This means the velocity of A as seen by an observer on B.

To view the situation from the point of view of an observer on B we need to imagine a velocity of $-\mathbf{v}_B$ is imposed on the whole system. It will then seem as though B is reduced to rest and the velocity A has in this system $(= \mathbf{v}_A - \mathbf{v}_B)$ will equal the velocity of A as seen by an observer on B. i.e. ${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B$.



(All velocities are relative to something. When we write $\mathbf{v}_A$ we usually mean the velocity of A relative to the earth.)

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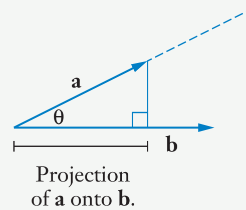
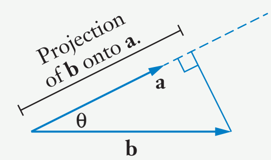
We define the **scalar product** of two vectors **a** and **b** to be the magnitude of **a** multiplied by the magnitude of **b** multiplied by the cosine of the angle between **a** and **b**. We write this product as **a**.**b** and say this as 'a dot b'. For this reason the scalar product is also referred to as the 'dot product'.

Thus $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ where θ is the angle between **a** and **b**.

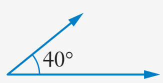
- '**a** cos θ' is the **projection** of **a** onto **b**, or the **resolved part** of **a** in the direction of **b**.

  Similarly we can refer to 'b cos θ' as the projection of **b** onto **a**, or the resolved part of **b** in the direction of **a**.

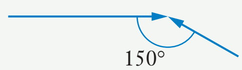




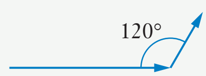
- Remember that the angle between two vectors refers to the angle between the vectors when they are either both directed away from a point or both directed towards it.



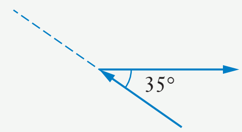
Angle between the vectors is 40°.



Angle between the vectors is 150°.



Angle between the vectors is 60°.



Angle between the vectors is 145°.

- It follows, from the definition of the scalar product of two vectors, that if **a** and **b** are perpendicular to each other then **a**.**b** = 0.
- It can be shown that the following algebraic properties of the scalar product follow from the definition of **a**.**b**.

$$\mathbf{a}.\mathbf{b} = \mathbf{b}.\mathbf{a} \qquad \mathbf{a}.(\lambda\mathbf{b}) = (\lambda\mathbf{a}).\mathbf{b} = \lambda(\mathbf{a}.\mathbf{b}) \qquad \mathbf{a}.\mathbf{a} = a^2$$

$$\mathbf{a}.(\mathbf{b}+\mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c} \qquad (\mathbf{a}+\mathbf{b}).\mathbf{c} = \mathbf{a}.\mathbf{c} + \mathbf{b}.\mathbf{c}$$

$$(\mathbf{a}+\mathbf{b}).(\mathbf{c}+\mathbf{d}) = \mathbf{a}.\mathbf{c} + \mathbf{a}.\mathbf{d} + \mathbf{b}.\mathbf{c} + \mathbf{b}.\mathbf{d}$$

- It also follows that:

  If $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$ then $\mathbf{a}.\mathbf{b} = x_1x_2 + y_1y_2$

  It is assumed that you are familiar with using the above statement to determine the angle between two vectors, given the vectors in component form.

It is assumed that from your previous study of vectors you are familiar with using the above ideas to prove various geometrical results.

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Complex numbers

You should be familiar with the idea of a **complex number** as one that is written in the form $a + bi$ (sometimes written $a + ib$) where a and b are real and $i = \sqrt{-1}$.

For some students the first encounter with i representing $\sqrt{-1}$ is when they use their calculator to solve a quadratic equation that has 'no real roots'. For example when solving the quadratic equation

$$x^2 + 6x + 13 = 0.$$

If we attempt to solve this equation using a calculator we may be given a message indicating there are no solutions, or we may be given solutions that involve 'i', dependent upon whether our calculator is set to solve for real solutions or complex solutions. See the displays below.

solve($x^2 + 6 \cdot x + 13 = 0, x$)
No Solution

solve($x^2 + 6 \cdot x + 13 = 0, x$)
$\{x = -3 - 2 \cdot i, x = -3 + 2 \cdot i\}$

The complex number $a + bi$ consists of a real part and an imaginary part.

For	$z = a + bi$	we say that the real part of z is a:	$\text{Re}(z) = a$
		and the imaginary part of z is b:	$\text{Im}(z) = b$.
Thus if	$z = 4 + 5i$	then	$\text{Re}(z) = 4$
		and	$\text{Im}(z) = 5$.

Complex numbers can be added to each other, subtracted from each other, multiplied together, divided by each other and multiplied or divided by a real number.

This is demonstrated below, and on the next page, for $w = 2 + 3i$ and $z = 5 - 4i$. Note that in each case the answer is given in the form $a + bi$ and especially note how this is achieved when one complex number is divided by another on the next page.

$$\begin{aligned} w + z &= (2 + 3i) + (5 - 4i) \\ &= 7 - i \end{aligned}$$

$$\begin{aligned} w - z &= (2 + 3i) - (5 - 4i) \\ &= 2 + 3i - 5 + 4i \\ &= -3 + 7i \end{aligned}$$

$$\begin{aligned} 3w - 2z &= 3(2 + 3i) - 2(5 - 4i) \\ &= 6 + 9i - 10 + 8i \\ &= -4 + 17i \end{aligned}$$

$$\begin{aligned} wz &= (2 + 3i)(5 - 4i) \\ &= 10 - 8i + 15i - 12i^2 \\ &= 10 - 8i + 15i + 12 \\ &= 22 + 7i \end{aligned}$$

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$$\begin{aligned} z^2 &= (5-4i)(5-4i) \\ &= 25-20i-20i+16i^2 \\ &= 25-20i-20i-16 \\ &= 9-40i \end{aligned}$$

$$\begin{aligned} \frac{w}{z} &= \frac{(2+3i)}{(5-4i)} \\ \therefore \quad \frac{w}{z} &= \frac{(2+3i)}{(5-4i)}\frac{(5+4i)}{(5+4i)} \\ &= \frac{10+8i+15i+12i^2}{25+20i-20i-16i^2} \\ &= \frac{-2+23i}{41} \\ &= -\frac{2}{41}+\frac{23}{41}i \end{aligned}$$

Alternatively these answers can be obtained from a calculator.

If $z = a + bi$ we say that $a - bi$ is the **conjugate** of z (also referred to as the **complex conjugate** of z). We use the symbol $\bar{z}$ for the conjugate of z.

Thus, if $z = 2 + 3i$ then $\bar{z} = 2 - 3i$,
if $z = 5 - 7i$ then $\bar{z} = 5 + 7i$,
if $z = -2 + 8i$ then $\bar{z} = -2 - 8i$,
if $z = -3 - 4i$ then $\bar{z} = -3 + 4i$, etc.

For any complex number z $(= a + bi)$, both the sum

$$\begin{aligned} & z + \bar{z} \\ =\; & (a + bi) + (a - bi) \\ =\; & 2a \end{aligned}$$

and the product

$$\begin{aligned} & z\bar{z} \\ =\; & (a + bi)(a - bi) \\ =\; & a^2 + b^2 \end{aligned}$$

are real.

(The fact that the product $z\bar{z}$ is real was used to produce a real denominator when $\frac{w}{z}$ was formed earlier on this page.)

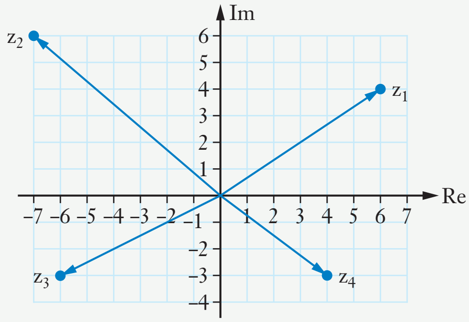
If two complex numbers, w and z, are equal then

$$\text{Re}(w) = \text{Re}(z) \qquad \textbf{and} \qquad \text{Im}(w) = \text{Im}(z).$$

If we write the complex number $a + bi$ as an **ordered pair** (a, b) this can be used like coordinates to represent the complex number as a point on a graph. Instead of x and y axes we have real and imaginary axes. Such a graphical representation is called an **Argand diagram** and the plane containing the real and imaginary axes is referred to as the **complex plane**. The complex number $a + bi$ can be thought of as the point (a, b) on the Argand diagram or as the vector from the origin to the point (a, b).

The Argand diagram on the right shows the complex numbers

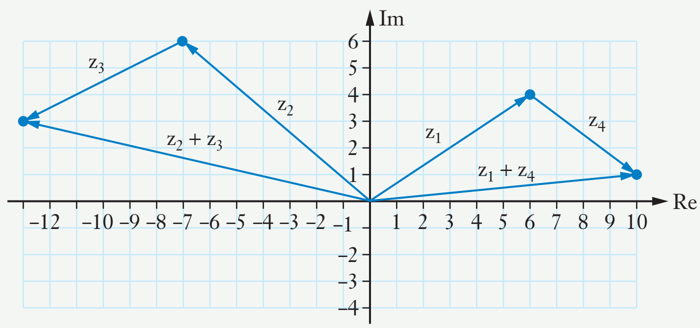
$z_1 = 6 + 4i,$
$z_2 = -7 + 6i,$
$z_3 = -6 - 3i,$
and $z_4 = 4 - 3i.$



Notice that we can then add complex numbers in the complex plane using the 'nose to tail' method of vector addition.

$$z_2 + z_3 = (-7 + 6i) + (-6 - 3i) = -13 + 3i$$

$$z_1 + z_4 = (6 + 4i) + (4 - 3i) = 10 + i$$



Circles

The reader is reminded of the following:

$x^2 + y^2 = r^2$ is the equation of a circle centre (0, 0) and radius r,

$(x - p)^2 + (y - q)^2 = r^2$ is the equation of a circle centre (p, q) and radius r.

Given the equation: $x^2 + y^2 + 6y = 10x$

Create gaps $x^2 - 10x + \ldots + y^2 + 6y + \ldots = 0$

Complete the squares $x^2 - 10x + 25 + y^2 + 6y + 9 = 0 + 25 + 9$

i.e. $(x - 5)^2 + (y + 3)^2 = 34$

Thus $x^2 + y^2 + 6y = 10x$ is the equation of circle, centre $(5, -3)$ and radius $\sqrt{34}$.

By way of practice confirm that the equation $2x^2 + x + 2y^2 - 5y = 3$ is that of a circle centre (–0.25, 1.25) and radius $\frac{5\sqrt{2}}{4}$. (Hint: Divide through by 2 first.)

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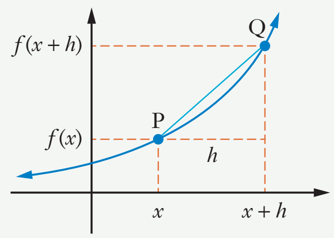
Differentiation

It is assumed that you are familiar with the idea of the **gradient**, or *slope*, of a line and, in particular, that whilst a straight line has the same gradient everywhere, the gradient of a curve varies as we move along the curve.



To find the gradient at a particular point, P, on a curve $y = f(x)$ we choose some other point, Q, on the curve whose x-coordinate is a little more than that of point P.

Suppose P has an x-coordinate of x and Q has an x-coordinate of $(x + h)$.

The corresponding y-coordinates of P and Q will then be $f(x)$ and $f(x + h)$.

Thus the gradient of PQ = $\dfrac{f(x+h)-f(x)}{h}$.

We then bring Q closer and closer to P, i.e. we allow h to tend to zero, and we determine the limiting value of the gradient of PQ.

i.e. Gradient at P = limit of $\dfrac{f(x+h)-f(x)}{h}$ as h tends to zero.

This gives us the **instantaneous rate of change** of the function at P.

The process of determining the **gradient formula** or **gradient function** of a curve is called **differentiation**.

Writing h, the small increase, or *increment*, in the x coordinate, as δx, where 'δ' is a Greek letter pronounced 'delta', and $f(x + h) - f(x)$, the small increment in the y coordinate as δy, we have:

$$\text{Gradient function} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

This **derivative** is written as $\dfrac{dy}{dx}$ and pronounced 'dee y by dee x'.

This 'limiting chord process' gives the following results:

If $y = x^2$ then $\dfrac{dy}{dx} = 2x$.

If $y = x^3$ then $\dfrac{dy}{dx} = 3x^2$.

If $y = x^4$ then $\dfrac{dy}{dx} = 4x^3$.

If $y = x^5$ then $\dfrac{dy}{dx} = 5x^4$.

The general statement is:

If $y = ax^n$ then $\dfrac{dy}{dx} = anx^{n-1}$.

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You should also be familiar with the following points:

- If $y = f(x)$ then the derivative of y with respect to x can be written as $\frac{dy}{dx}$, $\frac{df}{dx}$ or $\frac{d}{dx}f(x)$.
- A shorthand notation using a 'dash' may be used for differentiation with respect to x. Thus if $y = f(x)$ we can write $\frac{dy}{dx}$ as $f'(x)$ or simply y' or f'.
- If $y = f(x) \pm g(x)$ then $\frac{dy}{dx} = f'(x) \pm g'(x)$.

Whenever we are faced with the task of finding the gradient formula, gradient function or derivative of some 'new' function, for which we do not already have a rule, for example if we wanted to determine the gradient function for $y = \sin x$, we simply go back to the basic principle:

$$\text{Gradient at P}(x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Sketching graphs

The use of differentiation to determine the gradient function of a curve can give useful information about the nature of the graph and assist us in making a **sketch** of the graph. Whilst we would not expect to be able to read values from the sketch of a graph with any great accuracy, the sketch should still be neatly drawn and should show the noteworthy features of the graph. In relation to graph sketching it is assumed that you are already familiar with, and understand, the following terms:

Turning points.

Stationary points.

Local and global maxima and minima.

Horizontal, and non-horizontal, points of inflection.

Intercepts with the axes.

Asymptotes.

Concavity.

Symmetry.

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Antidifferentiation

Antidifferentiation is, as its name suggests, the opposite of differentiation.

Given the **derivative**, or **gradient function**, $\frac{dy}{dx}$, antidifferentiation returns us to the function, or **primitive**.

However there are a many functions that differentiate to $2x$, for example:

If $y = x^2$ then $\frac{dy}{dx} = 2x$.

If $y = x^2 + 1$ then $\frac{dy}{dx} = 2x$.

If $y = x^2 - 3$ then $\frac{dy}{dx} = 2x$. Etc.

Thus we say that the antiderivative of $2x$ is $x^2 + c$ where c is some constant. Given further information it may be possible to determine the value of this constant.

The general statement is: If $\frac{dy}{dx} = ax^n$ then $y = \frac{ax^{n+1}}{n+1} + c$

Remembered as: '*Increase the power by one and divide by the new power.*'

(Clearly this rule cannot apply for $n = -1$. Such a situation is beyond the scope of this unit.)

Hence the antiderivative of $6x^2 + 7$ is $\frac{6x^3}{3} + \frac{7x^1}{1} + c$

i.e $2x^3 + 7x + c$

It is also assumed that you are familiar with the fact that antidifferentiation is also known as **integration**. Instead of being asked to find the antiderivative of $6x^2 + 7$ we could be asked to **integrate** $6x^2 + 7$.

Integration uses the symbol $\int$.

Hence the fact that the antiderivative of $6x^2 + 7$ is $2x^3 + 7x + c$

could be written as $\int (6x^2 + 7)\,dx = 2x^3 + 7x + c$

the 'dx' indicating that the antidifferentiation, or integration, is with respect to the variable x.

Our general rule for antidifferentiating ax^n could then be written:

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$

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Trigonometrical identities

It is assumed that you are already familiar with the following trigonometrical identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{(Pythagorean identity.)}$$

(Sometimes referred to as the first of the Pythagorean identities.)

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

(From which $\sin 2A = 2\sin A\cos A$ and $\cos 2A = \cos^2 A - \sin^2 A$.)

It is also assumed that you are familiar with the fact that the reciprocals of $\sin\theta$, $\cos\theta$ and $\tan\theta$, i.e. $\frac{1}{\sin\theta}$, $\frac{1}{\cos\theta}$ and $\frac{1}{\tan\theta}$ are given names of their own.

$$\frac{1}{\cos\theta} = \sec\theta \qquad \frac{1}{\sin\theta} = \operatorname{cosec}\theta \qquad \frac{1}{\tan\theta} = \cot\theta \quad \left(= \frac{\cos\theta}{\sin\theta}\right)$$

sec, cosec and cot being abbreviations for secant, cosecant and cotangent.

Pythagorean identities can be established for these reciprocal functions:

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Matrices

It is assumed that you are familiar with the 'rows and columns' presentation of information called a **matrix**.

Whilst the following ideas should all be familiar to you:

the size of a matrix,
a row matrix,
a column matrix,
equal matrices,
addition and subtraction of matrices,
multiplication of a matrix by a scalar,
multiplication of a matrix by a matrix,
square matrices,
the leading diagonal of a square matrix,
multiplicative identity matrices,
and the inverse of a 2×2 matrix,

for this unit, two ideas of particular relevance are:

the determinant of a 2×2 matrix
and solving simultaneous equations by matrix methods.

You are reminded of these two ideas on the next page.

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- For the 2 × 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the quantity $(ad - bc)$ is the **determinant** of A.

- To solve the simultaneous equations $\begin{cases} x - y = 7 \\ 2x + 3y = 4 \end{cases}$

  First write the equations in matrix form: $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$.

  Then use the multiplicative inverse of $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, i.e. $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$, as shown below.

  Given $$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

  It follows that $$\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

  Hence $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$
  $$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 \\ 4 \end{bmatrix}$$
  $$= \frac{1}{5}\begin{bmatrix} 25 \\ -10 \end{bmatrix}$$

  Thus, $x = 5$ and $y = -2$.

Use of technology

You are encouraged to use your calculator, computer programs and the internet whenever appropriate during this unit.

However you should make sure that you can also perform the basic processes without the assistance of such technology when required to do so.

Note: The illustrations of calculator displays shown in the book may not exactly match the display from your calculator. The illustrations are not meant to show you exactly what your calculator will necessarily display but are included more to inform you that at that moment the use of a calculator could well be appropriate.

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